ENUMERATION OF SOME PARTICULAR $2N \times 10$ N-TIMES PERSYMMETRIC MATRICES OVER $\mathbb{F}_2$ BY RANK

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Résumé. Dans cet article nous comptons le nombre de certaines $2n \times 10$ n-fois matrices persymétriques de rang i sur $\mathbb{F}_2$.

Abstract. In this paper we count the number of some particular $2n \times$ 10 n-times persymmetric rank i matrices over $\mathbb{F}_2$.

Contents

1. Introduction.

In this paper we propose to compute the number $\Gamma_i^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times 10}$ of rank i $2n\times 10$ n-times persymmetric matrices over $\mathbb{F}_2$ of the below form for $0 \leqslant i \leqslant \inf(2n,10)$

$$
(1.1)\qquad \left(\begin{array}{cccccccccc}
\alpha_1^{(1)} & \alpha_2^{(1)} & \alpha_3^{(1)} & \alpha_4^{(1)} & \alpha_5^{(1)} & \alpha_6^{(1)} & \alpha_7^{(1)} & \alpha_8^{(1)} & \alpha_9^{(1)} & \alpha_{10}^{(1)} \\
\alpha_2^{(1)} & \alpha_3^{(1)} & \alpha_4^{(1)} & \alpha_5^{(1)} & \alpha_6^{(1)} & \alpha_7^{(1)} & \alpha_8^{(1)} & \alpha_9^{(1)} & \alpha_{10}^{(1)} & \alpha_{11}^{(1)} \\
\hline
\alpha_1^{(2)} & \alpha_2^{(2)} & \alpha_3^{(2)} & \alpha_4^{(2)} & \alpha_5^{(2)} & \alpha_6^{(2)} & \alpha_7^{(2)} & \alpha_8^{(2)} & \alpha_9^{(2)} & \alpha_{10}^{(2)} \\
\alpha_2^{(2)} & \alpha_3^{(2)} & \alpha_4^{(2)} & \alpha_5^{(2)} & \alpha_6^{(2)} & \alpha_7^{(2)} & \alpha_8^{(2)} & \alpha_9^{(2)} & \alpha_{10}^{(2)} & \alpha_{11}^{(2)} \\
\hline
\alpha_1^{(3)} & \alpha_2^{(3)} & \alpha_3^{(3)} & \alpha_4^{(3)} & \alpha_5^{(3)} & \alpha_6^{(3)} & \alpha_7^{(3)} & \alpha_8^{(3)} & \alpha_9^{(3)} & \alpha_{10}^{(3)} \\
\alpha_2^{(3)} & \alpha_3^{(3)} & \alpha_4^{(3)} & \alpha_5^{(3)} & \alpha_6^{(3)} & \alpha_7^{(3)} & \alpha_8^{(3)} & \alpha_9^{(3)} & \alpha_{10}^{(3)} & \alpha_{11}^{(3)} \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
\alpha_1^{(n)} & \alpha_2^{(n)} & \alpha_3^{(n)} & \alpha_4^{(n)} & \alpha_5^{(n)} & \alpha_6^{(n)} & \alpha_7^{(n)} & \alpha_8^{(n)} & \alpha_9^{(n)} & \alpha_{10}^{(n)} \\
\alpha_2^{(n)} & \alpha_3^{(n)} & \alpha_4^{(n)} & \alpha_5^{(n)} & \alpha_6^{(n)} & \alpha_7^{(n)} & \alpha_8^{(n)} & \alpha_9^{(n)} & \alpha_{10}^{(n)} & \alpha_{11}^{(n)}
\end{array}\right)
$$

We remark that the results in this paper are just a generalization of the results obtained in the author's paper [14].

2. Some notations concerning the field of Laurent Series $\mathbb{F}_2((T^{-1}))$.

We denote by $\mathbb{F}_2\big((T^{-1})\big) = \mathbb{K}$ the completion of the field $\mathbb{F}_2(T)$, the field of rational fonctions over the finite field $\mathbb{F}_2$, for the infinity valuation $\mathfrak{v} = \mathfrak{v}_\infty$ defined by $\mathfrak{v}\big(\frac{A}{B}\big) = degB - degA$ for each pair (A,B) of non-zero polynomials. Then every element non-zero t in $\mathbb{F}_2\big(\big(\frac{1}{T}\big)\big)$ can be expanded in a unique way in a convergent Laurent series $t = \sum_{j=-\infty}^{-\mathfrak{v}(t)} t_j T^j \; where \; t_j \in \mathbb{F}_2$.
We associate to the infinity valuation $\mathfrak{v} = \mathfrak{v}_\infty$ the absolute value $|\cdot|_\infty$ defined by

$$|t|_\infty = |t| = 2^{-\mathfrak{v}(t)}.$$

We denote E the Character of the additive locally compact group $\mathbb{F}_2\big(\big(\frac{1}{T}\big)\big)$ defined by

$$E\Big(\sum_{j=-\infty}^{-\mathfrak{v}(t)} t_j T^j\Big) = \begin{cases} 1 & \text{if } t_{-1} = 0, \\ -1 & \text{if } t_{-1} = 1. \end{cases}$$

We denote $\mathbb{P}$ the valuation ideal in $\mathbb{K}$, also denoted the unit interval of $\mathbb{K}$, i.e. the open ball of radius 1 about 0 or, alternatively, the set of all Laurent

series

$$\sum_{i\geq 1}\alpha_i T^{-i} \quad (\alpha_i \in \mathbb{F}_2)$$

and, for every rational integer j, we denote by $\mathbb{P}_j$ the ideal $\{t \in \mathbb{K}|\ \mathfrak{v}(t) > j\}$.
The sets $\mathbb{P}_j$ are compact subgroups of the additive locally compact group $\mathbb{K}$.
All $t \in \mathbb{F}_2\left(\left(\frac{1}{T}\right)\right)$ may be written in a unique way as $t = [t] + \{t\}$, $[t] \in \mathbb{F}_2[T],\ \{t\} \in \mathbb{P}(=\mathbb{P}_0)$.
We denote by dt the Haar measure on $\mathbb{K}$ chosen so that

$$\int_{\mathbb{P}} dt = 1.$$

$$Let \quad (t_1,t_2,\ldots,t_n) = \big(\sum_{j=-\infty}^{-\nu(t_1)}\alpha_j^{(1)}T^j, \sum_{j=-\infty}^{-\nu(t_2)}\alpha_j^{(2)}T^j,\ldots, \sum_{j=-\infty}^{-\nu(t_n)}\alpha_j^{(n)}T^j\big) \in \mathbb{K}^n.$$

We denote ψ the Character on $(\mathbb{K}^n,+)$ defined by

$$\begin{aligned}\psi\big(\sum_{j=-\infty}^{-\nu(t_1)}\alpha_j^{(1)}T^j, \sum_{j=-\infty}^{-\nu(t_2)}\alpha_j^{(2)}T^j,\ldots, \sum_{j=-\infty}^{-\nu(t_n)}\alpha_j^{(n)}T^j\big) &= E\big(\sum_{j=-\infty}^{-\nu(t_1)}\alpha_j^{(1)}T^j\big)\cdot E\big(\sum_{j=-\infty}^{-\nu(t_2)}\alpha_j^{(2)}T^j\big)\cdots E\big(\sum_{j=-\infty}^{-\nu(t_n)}\alpha_j^{(n)}T^j\big)\\ &= \begin{cases}1 & \text{if } \alpha_{-1}^{(1)}+\alpha_{-1}^{(2)}+\ldots+\alpha_{-1}^{(n)}=0\\ -1 & \text{if } \alpha_{-1}^{(1)}+\alpha_{-1}^{(2)}+\ldots+\alpha_{-1}^{(n)}=1\end{cases}\end{aligned}$$

3. Some results concerning n-times persymmetric matrices over $\mathbb{F}_2$.

$$Set \quad (t_1,t_2,\ldots,t_n) = \big(\sum_{i\geq 1}\alpha_i^{(1)}T^{-i}, \sum_{i\geq 1}\alpha_i^{(2)}T^{-i}, \sum_{i\geq 1}\alpha_i^{(3)}T^{-i},\ldots, \sum_{i\geq 1}\alpha_i^{(n)}T^{-i}\big) \in \mathbb{P}^n.$$

Denote by $D^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times k}(t_1,t_2,\ldots,t_n)$
the following $2n\times k$ n-times persymmetric matrix over the finite field $\mathbb{F}_2$.

$$(3.1)\qquad \left(\begin{array}{cccccccc}
\alpha_1^{(1)} & \alpha_2^{(1)} & \alpha_3^{(1)} & \alpha_4^{(1)} & \alpha_5^{(1)} & \alpha_6^{(1)} & \ldots & \alpha_k^{(1)} \\
\alpha_2^{(1)} & \alpha_3^{(1)} & \alpha_4^{(1)} & \alpha_5^{(1)} & \alpha_6^{(1)} & \alpha_7^{(1)} & \ldots & \alpha_{k+1}^{(1)} \\
\hline
\alpha_1^{(2)} & \alpha_2^{(2)} & \alpha_3^{(2)} & \alpha_4^{(2)} & \alpha_5^{(2)} & \alpha_6^{(2)} & \ldots & \alpha_k^{(2)} \\
\alpha_2^{(2)} & \alpha_3^{(2)} & \alpha_4^{(2)} & \alpha_5^{(2)} & \alpha_6^{(2)} & \alpha_7^{(2)} & \ldots & \alpha_{k+1}^{(2)} \\
\hline
\alpha_1^{(3)} & \alpha_2^{(3)} & \alpha_3^{(3)} & \alpha_4^{(3)} & \alpha_5^{(3)} & \alpha_6^{(3)} & \ldots & \alpha_k^{(3)} \\
\alpha_2^{(3)} & \alpha_3^{(3)} & \alpha_4^{(3)} & \alpha_5^{(3)} & \alpha_6^{(3)} & \alpha_7^{(3)} & \ldots & \alpha_{k+1}^{(3)} \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
\alpha_1^{(n)} & \alpha_2^{(n)} & \alpha_3^{(n)} & \alpha_4^{(n)} & \alpha_5^{(n)} & \alpha_6^{(n)} & \ldots & \alpha_k^{(n)} \\
\alpha_2^{(n)} & \alpha_3^{(n)} & \alpha_4^{(n)} & \alpha_5^{(n)} & \alpha_6^{(n)} & \alpha_7^{(n)} & \ldots & \alpha_{k+1}^{(n)}
\end{array}\right)$$

We denote by $\Gamma_i^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times k}$ the number of rank i n-times persymmetric matrices over $\mathbb{F}_2$ of the above form :

Let $f(t_1,t_2,\ldots,t_n)$ be the exponential sum in $\mathbb{P}^n$ defined by
$(t_1,t_2,\ldots,t_n)\in\mathbb{P}^n \longrightarrow$
$\displaystyle\sum_{degY\leq k-1}\sum_{degU_1\leq 1}E(t_1YU_1)\sum_{degU_2\leq 1}E(t_2YU_2)\ldots\sum_{degU_n\leq 1}E(t_nYU_n).$

Then

$$f_k(t_1,t_2,\ldots,t_n) = 2^{2n+k-rank\left[D^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times k}(t_1,t_2,\ldots,t_n)\right]}$$

Hence the number denoted by $R_{q,n}^{(k)}$ of solutions

$(Y_1,U_1^{(1)},U_2^{(1)},\ldots,U_n^{(1)},Y_2,U_1^{(2)},U_2^{(2)},\ldots,U_n^{(2)},\ldots Y_q,U_1^{(q)},U_2^{(q)},\ldots,U_n^{(q)})\in(\mathbb{F}_2[T])^{(n+1)q}$

of the polynomial equations

$$\begin{cases}
Y_1U_1^{(1)}+Y_2U_1^{(2)}+\ldots+Y_qU_1^{(q)}=0\\
Y_1U_2^{(1)}+Y_2U_2^{(2)}+\ldots+Y_qU_2^{(q)}=0\\
\vdots\\
Y_1U_n^{(1)}+Y_2U_n^{(2)}+\ldots+Y_qU_n^{(q)}=0
\end{cases}$$

$$\Leftrightarrow \begin{pmatrix} U_1^{(1)} & U_1^{(2)} & \ldots & U_1^{(q)} \\ U_2^{(1)} & U_2^{(2)} & \ldots & U_2^{(q)} \\ \vdots & \vdots & \vdots & \vdots \\ U_n^{(1)} & U_n^{(2)} & \ldots & U_n^{(q)} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

satisfying the degree conditions

$$degY_i \leq k-1, \quad degU_j^{(i)} \leq 1, \quad for \quad 1 \leq j \leq n, \quad 1 \leq i \leq q$$

is equal to the following integral over the unit interval in $\mathbb{K}^n$

$$\int_{\mathbb{P}^n} f_k^q(t_1, t_2, \ldots, t_n) dt_1 dt_2 \ldots dt_n.$$

Observing that $f(t_1, t_2, \ldots, t_n)$ is constant on cosets of $\prod_{j=1}^{n} \mathbb{P}_{k+1}$ in $\mathbb{P}^n$ the above integral is equal to

$$2^{q(2n+k)-(k+1)n} \sum_{i=0}^{\inf(2n,k)} \Gamma_i^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right] \times k} 2^{-iq} = R_{q,n}^{(k)}. \tag{3.2}$$

Recall that $R_{q,n}^{(k)}$ is equal to the number of solutions of the polynomial system

$$\begin{pmatrix} U_1^{(1)} & U_1^{(2)} & \ldots & U_1^{(q)} \\ U_2^{(1)} & U_2^{(2)} & \ldots & U_2^{(q)} \\ \vdots & \vdots & \vdots & \vdots \\ U_n^{(1)} & U_n^{(2)} & \ldots & U_n^{(q)} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.3}$$

satisfying the degree conditions

$$degY_i \leq k-1, \quad degU_j^{(i)} \leq 1, \quad for \quad 1 \leq j \leq n, \quad 1 \leq i \leq q.$$

From (3.2) we obtain for q = 1

$$2^{k-(k-1)n} \sum_{i=0}^{\inf(2n,k)} \Gamma_i^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right] \times k} 2^{-i} = R_{1,n}^{(k)} = 2^{2n} + 2^k - 1. \tag{3.4}$$

We have obviously

$$\sum_{i=0}^{k} \Gamma_i^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right] \times k} = 2^{(k+1)n}. \tag{3.5}$$

From the fact that the number of rank one persymmetric matrices over $\mathbb{F}_2$ is equal to three we obtain using combinatorial methods :

$$\Gamma_{1}^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right]\times k} = (2^n - 1)\cdot 3. \tag{3.6}$$

For more details see Cherly [11]

3.1. **Computation of $\Gamma_{7}^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right]\times k}$.** We recall (see section 3) that $\Gamma_{7}^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right]\times k}$ denotes the number of rank 7 n-times persymmetric matrices over $\mathbf{F}_2$ of the form (3.1) We shall need the following Lemma :

Lemma 3.1.
(3.7)

$$\Gamma_{7}^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right]\times k} = \begin{cases} 0 & \textit{if } n = 0, \\ 0 & \textit{if } n = 1, \\ 0 & \textit{if } n = 2, \\ 0 & \textit{if } n = 3, \\ 3720\cdot 2^{3k} - 416640\cdot 2^{2k} + 13332480\cdot 2^{k} - 121896960 & \textit{if } n = 4, \\ 115320\cdot [2^{3k} + 1148\cdot 2^{2k} - 2^{7}\cdot 917\cdot 2^{k} + 311\cdot 2^{13}] & \textit{if } n = 5, \end{cases}$$

(3.8)

$$\Gamma_{7}^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right]\times k} = \begin{cases} 255\cdot 2^{7n} - 381\cdot 2^{6n} - 31122\cdot 2^{5n} + 105648\cdot 2^{4n} & \\ +758880\cdot 2^{3n} - 4617984\cdot 2^{2n} + 7913472\cdot 2^{n} - 4128768 & \textit{if } k = 8, \\ 255\cdot 2^{7n} + 42291\cdot 2^{6n} - 219618\cdot 2^{5n} - 4053808\cdot 2^{4n} & \\ +32840160\cdot 2^{3n} - 82168576\cdot 2^{2n} + 81543168\cdot 2^{n} - 27983872 & \textit{if } k = 9, \end{cases}$$

Proof. Lemma 3.1 follows from Cherly[12,13 and 14]. $\square$

Lemma 3.2. *We postulate that :*

$$
\begin{aligned}
\Gamma_{7}^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times k} &= 255\cdot 2^{7n} + a(k)\cdot 2^{6n} + b(k)\cdot 2^{5n} + c(k)\cdot 2^{4n} \\
&\quad + d(k)\cdot 2^{3n} + e(k)\cdot 2^{2n} + f(k)\cdot 2^{n} + g(k) \\
&= 255\cdot 2^{7n} + [\frac{2667}{16}\cdot 2^{k} - 43053]\cdot 2^{6n} \\
&\quad + [\frac{465}{32}\cdot 2^{2k} - \frac{190341}{16}\cdot 2^{k} + 2062014]\cdot 2^{5n} \\
&\quad + [\frac{31}{168}\cdot 2^{3k} - \frac{45229}{96}\cdot 2^{2k} + \frac{6262403}{24}\cdot 2^{k} - \frac{817168432}{21}]\cdot 2^{4n} \\
&\quad + [-\frac{465}{168}\cdot 2^{3k} + \frac{231105}{48}\cdot 2^{2k} - \frac{4605205}{2}\cdot 2^{k} + \frac{2247886880}{7}]\cdot 2^{3n} \\
&\quad + [\frac{155}{12}\cdot 2^{3k} - \frac{233585}{12}\cdot 2^{2k} + \frac{26162884}{3}\cdot 2^{k} - \frac{3534612736}{3}]\cdot 2^{2n} \\
&\quad + [-\frac{155}{7}\cdot 2^{3k} + 31310\cdot 2^{2k} - 13600384\cdot 2^{k} + \frac{1466315\cdot 2^{13}}{7} + 11373\cdot 2^{13}]\cdot 2^{n} \\
&\quad + \frac{248}{21}\cdot 2^{3k} - \frac{48608}{3}\cdot 2^{2k} + \frac{20798464}{3}\cdot 2^{k} - \frac{293263\cdot 2^{16}}{21} \\
&= \frac{31}{168}\cdot [2^{4n} - 15\cdot 2^{3n} + 70\cdot 2^{2n} - 120\cdot 2^{n} + 64]\cdot 2^{3k} \\
&\quad + \frac{1}{96}\cdot [1395\cdot 2^{5n} - 45229\cdot 2^{4n} + 462210\cdot 2^{3n} - 1868680\cdot 2^{2n} + 3005760\cdot 2^{n} - 1555456]\cdot 2^{2k} \\
&\quad + \frac{1}{48}\cdot [8001\cdot 2^{6n} - 571023\cdot 2^{5n} + 12524806\cdot 2^{4n} - 110524920\cdot 2^{3n} + 418606144\cdot 2^{2n} \\
&\quad - 652818432\cdot 2^{n} + 332775424]\cdot 2^{k} \\
&\quad + \frac{1}{21}\cdot [5355\cdot 2^{7n} - 904113\cdot 2^{6n} + 43302294\cdot 2^{5n} - 817168432\cdot 2^{4n} + 6743660640\cdot 2^{3n} \\
&\quad - 96649567\cdot 2^{8}\cdot 2^{2n} + 4637778\cdot 2^{13}\cdot 2^{n} - 293263\cdot 2^{16}]
\end{aligned} \tag{3.9}
$$

Proof.

From the expression of $\Gamma_{7}^{\left[\substack{2\\ \vdots\\ 2}\right]\times k}$ in (3.8) for k=8,9 we assume that $\Gamma_{7}^{\left[\substack{2\\ \vdots\\ 2}\right]\times k}$ can be written in the form :

$$255\cdot 2^{7n}+a(k)\cdot 2^{6n}+b(k)\cdot 2^{5n}+c(k)\cdot 2^{4n}$$
$$+d(k)\cdot 2^{3n}+e(k)\cdot 2^{2n}+f(k)\cdot 2^{n}+g(k)$$

Set $Y=2^n,$ then $\Gamma_{7}^{\left[\substack{2\\ \vdots\\ 2}\right]\times k}$

$$=255\cdot Y^7+a(k)\cdot Y^6+b(k)\cdot Y^5+c(k)\cdot Y^4+d(k)\cdot Y^3+e(k)\cdot Y^2+f(k)\cdot Y+g(k)$$

$$\Gamma_{7}^{\left[\substack{2\\ \vdots\\ 2}\right]\times k}=0 \quad \text{for } n\in\{0,1,2,3\}$$

Then $\Gamma_{7}^{\left[\substack{2\\ \vdots\\ 2}\right]\times k}=(Y-1)(Y-2)(Y-4)(Y-8)[255\cdot Y^3+\alpha(k)\cdot Y^2+\beta(k)\cdot Y+\gamma(k)]$

$$=[2^{4n}-15\cdot 2^{3n}+70\cdot 2^{2n}-120\cdot 2^{n}+64]\cdot[255\cdot 2^{3n}+\alpha(k)\cdot 2^{2n}+\beta(k)\cdot 2^{n}+\gamma(k)]$$
$$=255\cdot 2^{7n}+(\alpha(k)-3825)\cdot 2^{6n}+(\beta(k)-15\cdot\alpha(k)+17850)\cdot 2^{5n}$$
$$+(\gamma(k)-15\cdot\beta(k)+70\cdot\alpha(k)-30600)\cdot 2^{4n}$$
$$+(-15\cdot\gamma(k)+70\cdot\beta(k)-120\cdot\alpha(k)+16320)\cdot 2^{3n}$$
$$+(70\cdot\gamma(k)-120\cdot\beta(k)+64\cdot\alpha(k))\cdot 2^{2n}$$
$$+(-120\cdot\gamma(k)+64\cdot\beta(k))\cdot 2^{n}+64\cdot\gamma(k)$$

We then deduce :

$$\begin{cases} a(k)=\alpha(k)-3825\\ b(k)=\beta(k)-15\cdot\alpha(k)+17850\\ c(k)=\gamma(k)-15\cdot\beta(k)+70\cdot\alpha(k)-30600\\ d(k)=-15\cdot\gamma(k)+70\cdot\beta(k)-120\cdot\alpha(k)+16320\\ e(k)=70\cdot\gamma(k)-120\cdot\beta(k)+64\cdot\alpha(k)\\ f(k)=-120\cdot\gamma(k)+64\cdot\beta(k)\\ g(k)=64\cdot\gamma(k)\end{cases} \tag{3.10}$$

To compute the expression of $\Gamma_{7}^{\left[\substack{2\\ \vdots\\ 2}\right]\times k}$ we need only to compute $\alpha(k),\beta(k)$ and $\gamma(k)$.

Computation of $\alpha(k)$

From the expressions of $\Gamma_{i}^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times k}$ for $2 \leqslant i \leqslant 6$ in **Lemma 3.3** [12] we assume that a(k) can be written in the form $a \cdot 2^{k} + b$.
We obtain from (3.8)

$$\begin{cases} a(k) = a \cdot 2^{k} + b \\ a(8) = a \cdot 256 + b = -381 \\ a(9) = a \cdot 512 + b = 42291 \\ a = \frac{2667}{16} \\ b = -43053 \\ (1) \quad \alpha(k) = a \cdot 2^{k} + b + 3825 = \frac{2667}{16} \cdot 2^{k} - 39228 \end{cases}$$

The case n=4.

$\Gamma_{7}^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times k} = (2^4-1)(2^4-2)(2^4-4)(2^4-8)[255\cdot 2^{12}+\alpha(k)\cdot 2^{8}+\beta(k)\cdot 2^{4}+\gamma(k)]$
$= 20160 \cdot [255 \cdot 2^{12} + \alpha(k) \cdot 2^{8} + \beta(k) \cdot 2^{4} + \gamma(k)] = 3720 \cdot 2^{3k} - 416640 \cdot 2^{2k} + 13332480 \cdot 2^{k} - 121896960$
$\Rightarrow 255 \cdot 2^{12} + \alpha(k) \cdot 2^{8} + \beta(k) \cdot 2^{4} + \gamma(k) = \frac{1}{20160} \cdot [3720 \cdot 2^{3k} - 416640 \cdot 2^{2k} + 13332480 \cdot 2^{k} - 121896960]$
$\Rightarrow (2) \quad 256 \cdot \alpha(k) + 16 \cdot \beta(k) + \gamma(k) = \frac{1}{20160} \cdot [3720 \cdot 2^{3k} - 416640 \cdot 2^{2k} + 13332480 \cdot 2^{k} - 121896960] - 255 \cdot 2^{12}$
$\Rightarrow (2) \quad 256 \cdot \alpha(k) + 16 \cdot \beta(k) + \gamma(k) = \frac{31}{168} \cdot [2^{3k} - 112 \cdot 2^{2k} + 3584 \cdot 2^{k} - 32768] - 255 \cdot 2^{12}$

The case n=5.

$\Gamma_{7}^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times k} = (2^5-1)(2^5-2)(2^5-4)(2^5-8)[255\cdot 2^{15}+\alpha(k)\cdot 2^{10}+\beta(k)\cdot 2^{5}+\gamma(k)]$
$= 624960 \cdot [255 \cdot 2^{15} + \alpha(k) \cdot 2^{10} + \beta(k) \cdot 2^{5} + \gamma(k)] = 115320 \cdot [2^{3k} + 1148 \cdot 2^{2k} - 2^{7} \cdot 917 \cdot 2^{k} + 311 \cdot 2^{13}]$
$\Rightarrow 255 \cdot 2^{15} + \alpha(k) \cdot 2^{10} + \beta(k) \cdot 2^{5} + \gamma(k) = \frac{1}{624960} \cdot [115320 \cdot (2^{3k} + 1148 \cdot 2^{2k} - 2^{7} \cdot 917 \cdot 2^{k} + 311 \cdot 2^{13})]$
$\Rightarrow (3) \quad 1024 \cdot \alpha(k) + 32 \cdot \beta(k) + \gamma(k) = \frac{1}{624960} \cdot [115320 \cdot (2^{3k} + 1148 \cdot 2^{2k} - 2^{7} \cdot 917 \cdot 2^{k} + 311 \cdot 2^{13})] - 255 \cdot 2^{15}$
$\Rightarrow (3) \quad 1024 \cdot \alpha(k) + 32 \cdot \beta(k) + \gamma(k) = \frac{31}{168} \cdot [2^{3k} + 1148 \cdot 2^{2k} - 117376 \cdot 2^{k} + 2547712] - 255 \cdot 2^{15}$
We then obtain :

$$(3.11)\quad \begin{cases}(1) \quad \alpha(k) = \frac{2667}{16}\cdot 2^{k}-39228\\ (2) \quad 256\cdot\alpha(k)+16\cdot\beta(k)+\gamma(k)\\ = \frac{31}{168}\cdot[2^{3k}-112\cdot 2^{2k}+3584\cdot 2^{k}-32768]-255\cdot 2^{12}\\ (3) \quad 1024\cdot\alpha(k)+32\cdot\beta(k)+\gamma(k)\\ = \frac{31}{168}\cdot[2^{3k}+1148\cdot 2^{2k}-117376\cdot 2^{k}+2547712]-255\cdot 2^{15}\end{cases}$$

From (3.11) we deduce :

$$(3.12)\quad \begin{cases}\alpha(k) = \frac{2667}{16}\cdot 2^{k}-39228\\ \beta(k) = \frac{465}{32}\cdot 2^{2k}-9396\cdot 2^{k}+1455744\\ \gamma(k) = \frac{31}{168}\cdot 2^{3k}-\frac{1519}{6}\cdot 2^{2k}+\frac{324976}{3}\cdot 2^{k}-\frac{300301312}{21}\end{cases}$$

Combining (3.10) and (3.12) we get :

(3.13)

$$\begin{cases}
a(k) = \alpha(k) - 3825 = \dfrac{2667}{16} \cdot 2^k - 43053 \\
b(k) = \beta(k) - 15 \cdot \alpha(k) + 17850 = \\
\frac{465}{32} \cdot 2^{2k} - 9396 \cdot 2^k + 1455744 - 15 \cdot [\frac{2667}{16} \cdot 2^k - 39228] + 17850 \\
= \frac{465}{32} \cdot 2^{2k} - \frac{190341}{16} \cdot 2^k + 2062014 \\
c(k) = \gamma(k) - 15 \cdot \beta(k) + 70 \cdot \alpha(k) - 30600 \\
= \frac{31}{168} \cdot 2^{3k} - \frac{1519}{6} \cdot 2^{2k} + \frac{324976}{3} \cdot 2^k - \frac{300301312}{21} \\
-15 \cdot [\frac{465}{32} \cdot 2^{2k} - 9396 \cdot 2^k + 1455744] + 70 \cdot [\frac{2667}{16} \cdot 2^k - 39228] - 30600 \\
= \frac{31}{168} \cdot 2^{3k} - \frac{45229}{96} \cdot 2^{2k} + \frac{6262403}{24} \cdot 2^k - \frac{817168432}{21} \\
d(k) = -15 \cdot \gamma(k) + 70 \cdot \beta(k) - 120 \cdot \alpha(k) + 16320 \\
= -15 \cdot [\frac{31}{168} \cdot 2^{3k} - \frac{1519}{6} \cdot 2^{2k} + \frac{324976}{3} \cdot 2^k - \frac{300301312}{21}] \\
+70 \cdot [\frac{465}{32} \cdot 2^{2k} - 9396 \cdot 2^k + 1455744] - 120 \cdot [\frac{2667}{16} \cdot 2^k - 39228] + 16320 \\
= -\frac{465}{168} \cdot 2^{3k} + \frac{231105}{48} \cdot 2^{2k} - \frac{4605205}{2} \cdot 2^k + \frac{2247886880}{7} \\
e(k) = 70 \cdot \gamma(k) - 120 \cdot \beta(k) + 64 \cdot \alpha(k) \\
= 70 \cdot [\frac{31}{168} \cdot 2^{3k} - \frac{1519}{6} \cdot 2^{2k} + \frac{324976}{3} \cdot 2^k - \frac{300301312}{21}] \\
-120 \cdot [\frac{465}{32} \cdot 2^{2k} - 9396 \cdot 2^k + 1455744] + 64 \cdot [\frac{2667}{16} \cdot 2^k - 39228] \\
= \frac{155}{12} \cdot 2^{3k} - \frac{233585}{12} \cdot 2^{2k} + \frac{26162884}{3} \cdot 2^k - \frac{3534612736}{3} \\
f(k) = -120 \cdot \gamma(k) + 64 \cdot \beta(k) \\
= -120 \cdot [\frac{31}{168} \cdot 2^{3k} - \frac{1519}{6} \cdot 2^{2k} + \frac{324976}{3} \cdot 2^k - \frac{300301312}{21}] \\
+64 \cdot [\frac{465}{32} \cdot 2^{2k} - 9396 \cdot 2^k + 1455744] \\
= -\frac{155}{7} \cdot 2^{3k} + 31310 \cdot 2^{2k} - 13600384 \cdot 2^k + \frac{12012052480}{7} + 93167616 \\
= -\frac{155}{7} \cdot 2^{3k} + 31310 \cdot 2^{2k} - 13600384 \cdot 2^k + \frac{1466315 \cdot 2^{13}}{7} + 11373 \cdot 2^{13} \\
g(k) = 64 \cdot \gamma(k) \\
= 64 \cdot [\frac{31}{168} \cdot 2^{3k} - \frac{1519}{6} \cdot 2^{2k} + \frac{324976}{3} \cdot 2^k - \frac{300301312}{21}] \\
= \frac{248}{21} \cdot 2^{3k} - \frac{48608}{3} \cdot 2^{2k} + \frac{20798464}{3} \cdot 2^k - \frac{293263 \cdot 2^{16}}{21}
\end{cases}$$

$\square$

3.2. **Computation of** $\Gamma_i^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right] \times 10}$ $for\ 0 \leqslant i \leqslant \inf(2n, 10)$**.** We shall need the following Lemma :

Lemma 3.3.

(3.14)

$$\begin{cases}
\Gamma_0^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times k} = 1 \quad \textit{if} \quad k \geqslant 1 \\
\Gamma_1^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times k} = (2^n-1)\cdot 3 \quad \textit{if} \quad k \geqslant 2 \\
\Gamma_2^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times k} = 7\cdot 2^{2n} + (2^{k+1}-25)\cdot 2^n - 2^{k+1} + 18 \quad \textit{for} \quad k \geqslant 3 \\
\Gamma_3^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times k} = 15\cdot 2^{3n} + (7\cdot 2^k - 133)\cdot 2^{2n} + (294 - 21\cdot 2^k)\cdot 2^n - 176 + 14\cdot 2^k \quad \textit{for} \quad k \geqslant 4 \\
\Gamma_4^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times k} = 31\cdot 2^{4n} + \dfrac{35\cdot 2^k - 1210}{2}\cdot 2^{3n} + \dfrac{2^{2k+2} - 783\cdot 2^k + 19028}{6}\cdot 2^{2n} \\
+(-2^{2k+1} + 269\cdot 2^k - 5744)\cdot 2^n + \dfrac{2^{2k+2} - 117\cdot 2^{k+2} + 9440}{3} \quad \textit{for} \quad k \geqslant 5 \\
\Gamma_5^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times k} = 63\cdot 2^{5n} + (\dfrac{155}{4}\cdot 2^k - 2573)\cdot 2^{4n} + (\dfrac{5}{2}\cdot 2^{2k} - \dfrac{2565}{4}\cdot 2^k + 29150)\cdot 2^{3n} \\
+\dfrac{1}{2}\cdot(-35\cdot 2^{2k} + 6265\cdot 2^k - 247520)\cdot 2^{2n} + (35\cdot 2^{2k} - 5490\cdot 2^k + 203872)\cdot 2^n \\
-20\cdot 2^{2k} + 2960\cdot 2^k - 106752 \quad \textit{for} \quad k \geqslant 6 \\
\Gamma_6^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times k} = 127\cdot 2^{6n} + [651\cdot 2^{k-3} - 10605]\cdot 2^{5n} + [\dfrac{155}{3}\cdot 2^{2k-3} - 22661\cdot 2^{k-3} + \dfrac{748154}{3}]\cdot 2^{4n} \\
+\frac{1}{168}\cdot[2^{3k+3} - 16723\cdot 2^{2k} + 5026378\cdot 2^k - 382091648]\cdot 2^{3n} \\
+[-\frac{1}{3}\cdot 2^{3k} + \frac{5649}{12}\cdot 2^{2k} - \frac{368711}{3}\cdot 2^k + 8753120]\cdot 2^{2n} \\
+[\frac{2}{3}\cdot 2^{3k} - \frac{2437}{3}\cdot 2^{2k} + \frac{597736}{3}\cdot 2^k - \frac{41276672}{3}]\cdot 2^n \\
-8\cdot[\frac{1}{21}\cdot 2^{3k} - \frac{163}{3}\cdot 2^{2k} + \frac{38816}{3}\cdot 2^k - \frac{18483200}{21}] \quad \textit{for} \quad k \geqslant 7 \\
\Gamma_7^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times k} = 255\cdot 2^{7n} + a(k)\cdot 2^{6n} + b(k)\cdot 2^{5n} + c(k)\cdot 2^{4n} + d(k)\cdot 2^{3n} + e(k)\cdot 2^{2n} + f(k)\cdot 2^n + g(k) \\
= 255\cdot 2^{7n} + [\frac{2667}{16}\cdot 2^k - 43053]\cdot 2^{6n} + [\frac{465}{32}\cdot 2^{2k} - \frac{190341}{16}\cdot 2^k + 2062014]\cdot 2^{5n} \\
+[\frac{31}{168}\cdot 2^{3k} - \frac{45229}{96}\cdot 2^{2k} + \frac{6262403}{24}\cdot 2^k - \frac{817168432}{21}]\cdot 2^{4n} \\
+[-\frac{465}{168}\cdot 2^{3k} + \frac{231105}{48}\cdot 2^{2k} - \frac{4605205}{2}\cdot 2^k + \frac{2247886880}{7}]\cdot 2^{3n} \\
+[\frac{155}{12}\cdot 2^{3k} - \frac{233585}{12}\cdot 2^{2k} + \frac{26162884}{3}\cdot 2^k - \frac{3534612736}{3}]\cdot 2^{2n} \\
+[-\frac{155}{7}\cdot 2^{3k} + 31310\cdot 2^{2k} - 13600384\cdot 2^k + \frac{1466315\cdot 2^{13}}{7} + 11373\cdot 2^{13}]\cdot 2^n \\
+\frac{248}{21}\cdot 2^{3k} - \frac{48608}{3}\cdot 2^{2k} + \frac{20798464}{3}\cdot 2^k - \frac{293263\cdot 2^{16}}{21} \quad \textit{for} \quad k \geqslant 8
\end{cases}$$

(3.15)
$$\begin{cases}\sum_{i=0}^{\inf(2n,k)}\Gamma_{i}^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times k} = 2^{(k+1)n},\\ \sum_{i=0}^{\inf(2n,k)}\Gamma_{i}^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times k}2^{-i} = 2^{n+k(n-1)}+2^{(k-1)n}-2^{(k-1)n-k},\\ \sum_{i=0}^{\inf(2n,k)}\Gamma_{i}^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times k}2^{-2i} = 2^{n+k(n-2)}+2^{-n+k(n-2)}\cdot[3\cdot 2^{k}-3]+2^{-2n+k(n-2)}\cdot[6\cdot 2^{k-1}-6]\\ +2^{-3n+kn}-6\cdot 2^{n(k-3)-k}+8\cdot 2^{-3n+k(n-2)}.\end{cases}$$

Proof. Lemma 3.3 follows from Lemma 3.3 in Cherly [14] and (3.9) □

We deduce from (3.14) and (3.15) with k=10.
(3.16)
$$\Gamma_{i}^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times 10} = \begin{cases}1 & \text{if } i=0,\\ (2^{n}-1)\cdot 3 & \text{if } i=1,\\ 7\cdot 2^{2n}+2023\cdot 2^{n}-2030 & \text{if } i=2,\\ 15\cdot 2^{3n}+7035\cdot 2^{2n}-21210\cdot 2^{n}+14160 & \text{if } i=3,\\ 31\cdot 2^{4n}+17315\cdot 2^{3n}+568590\cdot 2^{2n}-1827440\cdot 2^{n}+1241504 & \text{if } i=4,\\ 63\cdot 2^{5n}+37107\cdot 2^{4n}+1993950\cdot 2^{3n} & \\ -15266160\cdot 2^{2n}+31282272\cdot 2^{n}-18047232 & \text{if } i=5,\\ 127\cdot 2^{6n}+72723\cdot 2^{5n}+4120830\cdot 2^{4n}-24883824\cdot 2^{3n} & \\ +18602976\cdot 2^{2n}+54302976\cdot 2^{n}-52215808 & \text{if } i=6,\\ 255\cdot 2^{7n}+127635\cdot 2^{6n}+5117310\cdot 2^{5n}-67607280\cdot 2^{4n} & \\ +39863520\cdot 2^{3n}+1210256640\cdot 2^{2n}-3062415360\cdot 2^{n}+1874657280 & \text{if } i=7,\\ 511\cdot 2^{8n}+a_{7}^{(8)}\cdot 2^{7n}+a_{6}^{(8)}\cdot 2^{6n}+a_{5}^{(8)}\cdot 2^{5n}+a_{4}^{(8)}\cdot 2^{4n} & \\ +a_{3}^{(8)}\cdot 2^{3n}+a_{2}^{(8)}\cdot 2^{2n}+a_{1}^{(8)}\cdot 2^{n}+a_{0}^{(8)} & \text{if } i=8,\\ 1023\cdot 2^{9n}+a_{8}^{(9)}\cdot 2^{8n}+a_{7}^{(9)}\cdot 2^{7n}+a_{6}^{(9)}\cdot 2^{6n}+a_{5}^{(9)}\cdot 2^{5n}+a_{4}^{(9)}\cdot 2^{4n} & \\ +a_{3}^{(9)}\cdot 2^{3n}+a_{2}^{(9)}\cdot 2^{2n}+a_{1}^{(9)}\cdot 2^{n}+a_{0}^{(9)} & \text{if } i=9.\\ 2^{11n}-1023\cdot 2^{9n}+a_{8}^{(10)}\cdot 2^{8n}+a_{7}^{(10)}\cdot 2^{7n}+a_{6}^{(10)}\cdot 2^{6n}+a_{5}^{(10)}\cdot 2^{5n}+a_{4}^{(10)}\cdot 2^{4n} & \\ +a_{3}^{(10)}\cdot 2^{3n}+a_{2}^{(10)}\cdot 2^{2n}+a_{1}^{(10)}\cdot 2^{n}+a_{0}^{(10)} & \text{if } i=10.\end{cases}$$

where,

(3.17)
$$\begin{cases}\displaystyle\sum_{i=0}^{10}\Gamma_{i}^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times 10} = 2^{11n},\\ \displaystyle\sum_{i=0}^{10}\Gamma_{i}^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times 10}2^{10-i} = 2^{11n}+1023\cdot 2^{9n},\\ \displaystyle\sum_{i=0}^{10}\Gamma_{i}^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times 10}2^{20-2i} = 2^{11n}+3069\cdot 2^{9n}+3066\cdot 2^{8n}+1042440\cdot 2^{7n}.\end{cases}$$

Combining (3.16) and (3.17) we compute $a_i^{(j)}$ in (3.16) for $8\leqslant j\leqslant 10,\ 0\leqslant i\leqslant j-1$
and we obtain from (3.16)

(3.18)
$$\Gamma_{i}^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times 10} = \begin{cases}1 & \text{if } i=0,\\ (2^n-1)\cdot 3 & \text{if } i=1,\\ 7\cdot 2^{2n}+2023\cdot 2^n-2030 & \text{if } i=2,\\ 15\cdot 2^{3n}+7035\cdot 2^{2n}-21210\cdot 2^n+14160 & \text{if } i=3,\\ 31\cdot 2^{4n}+17315\cdot 2^{3n}+568590\cdot 2^{2n}-1827440\cdot 2^n+1241504 & \text{if } i=4,\\ 63\cdot 2^{5n}+37107\cdot 2^{4n}+1993950\cdot 2^{3n} & \\ -15266160\cdot 2^{2n}+31282272\cdot 2^n-18047232 & \text{if } i=5,\\ 127\cdot 2^{6n}+72723\cdot 2^{5n}+4120830\cdot 2^{4n}-24883824\cdot 2^{3n} & \\ +18602976\cdot 2^{2n}+54302976\cdot 2^n-52215808 & \text{if } i=6,\\ 255\cdot 2^{7n}+127635\cdot 2^{6n}+5117310\cdot 2^{5n} & \\ -67607280\cdot 2^{4n}+39863520\cdot 2^{3n}+1210256640\cdot 2^{2n} & \\ -3062415360\cdot 2^n+1874657280 & \text{if } i=7,\\ 511\cdot 2^{8n}+171955\cdot 2^{7n}-897890\cdot 2^{6n}-38376240\cdot 2^{5n} & \\ +323250144\cdot 2^{4n}+271514880\cdot 2^{3n}-436135\cdot 2^{14}\cdot 2^{2n} & \\ +242795\cdot 2^{16}\cdot 2^n-4445\cdot 2^{21} & \text{if } i=8,\\ 1023\cdot 2^{9n}-1533\cdot 2^{8n}-517650\cdot 2^{7n}+1798320\cdot 2^{6n} & \\ +78214752\cdot 2^{5n}-559464192\cdot 2^{4n}-783237120\cdot 2^{3n} & \\ +200235\cdot 2^{16}\cdot 2^{2n}-106680\cdot 2^{18}\cdot 2^n+480\cdot 2^{25} & \text{if } i=9.\\ 2^{11n}-1023\cdot 2^{9n}+1022\cdot 2^{8n}+345440\cdot 2^{7n}-1028192\cdot 2^{6n} & \\ -45028608\cdot 2^{5n}+299663360\cdot 2^{4n}+494731264\cdot 2^{3n} & \\ -27432\cdot 2^{18}\cdot 2^{2n}+57344\cdot 2^{18}\cdot 2^n-256\cdot 2^{25} & \text{if } i=10.\end{cases}$$

Example. **Computation of $R_{q,n}^{(k)}$ in the case k=10, q=4 (see** (3.2) **and** (3.3)**)** The number denoted by $R_{4,n}^{(10)}$ of solutions

$(Y_1, U_1^{(1)}, U_2^{(1)}, \ldots, U_n^{(1)}, Y_2, U_1^{(2)}, U_2^{(2)}, \ldots, U_n^{(2)}, Y_3, U_1^{(3)}, U_2^{(3)}, \ldots, U_n^{(3)}, Y_4, U_1^{(4)}, U_2^{(4)}, \ldots, U_n^{(4)}) \in (\mathbb{F}_2[T])^{4(n+1)}$

of the polynomial equations

$$\begin{cases} Y_1U_1^{(1)} + Y_2U_1^{(2)} + Y_3U_1^{(3)} + Y_4U_1^{(4)} = 0 \\ Y_1U_2^{(1)} + Y_2U_2^{(2)} + Y_3U_2^{(3)} + Y_4U_2^{(4)} = 0 \\ \vdots \\ Y_1U_n^{(1)} + Y_2U_n^{(2)} + Y_3U_n^{(3)} + Y_4U_n^{(4)} = 0 \end{cases}$$

satisfying the degree conditions

$$degY_i \leq 9, \quad degU_j^{(i)} \leq 1, \quad for \quad 1 \leq j \leq n, \quad 1 \leq i \leq 4$$

is equal to

$$R_{q,n}^{(k)} = 2^{q(2n+k)-(k+1)n} \sum_{i=0}^{\inf(2n,k)} \Gamma_i^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right] \times k} 2^{-iq} = R_{4,n}^{(10)} = 2^{40-3n} \sum_{i=0}^{10} \Gamma_i^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right] \times 10} 2^{-i4}$$

$$\begin{aligned}
&= 2^{40-3n} \cdot \Big[2^{-40} \cdot 2^{11n} \\
&+(-1023 \cdot 2^{-40} + 1023 \cdot 2^{-36}) \cdot 2^{9n} \\
&+(1022 \cdot 2^{-40} - 1533 \cdot 2^{-36} + 511 \cdot 2^{-32}) \cdot 2^{8n} \\
&+(345440 \cdot 2^{-40} - 517650 \cdot 2^{-36} + 171955 \cdot 2^{-32} + 255 \cdot 2^{-28}) \cdot 2^{7n} \\
&+(-1028192 \cdot 2^{-40} + 1798320 \cdot 2^{-36} - 897890 \cdot 2^{-32} \\
&+127635 \cdot 2^{-28} + 127 \cdot 2^{-24}) \cdot 2^{6n} \\
&+(-45028608 \cdot 2^{-40} + 78214752 \cdot 2^{-36} - 38376240 \cdot 2^{-32} \\
&+5117310 \cdot 2^{-28} + 72723 \cdot 2^{-24} + 63 \cdot 2^{-20}) \cdot 2^{5n} \\
&+(299663360 \cdot 2^{-40} - 559464192 \cdot 2^{-36} + 323250144 \cdot 2^{-32} \\
&-67607280 \cdot 2^{-28} + 4120830 \cdot 2^{-24} + 37107 \cdot 2^{-20} + 31 \cdot 2^{-16}) \cdot 2^{4n} \\
&+(494731264 \cdot 2^{-40} - 783237120 \cdot 2^{-36} + 271514880 \cdot 2^{-32} \\
&39863520 \cdot 2^{-28} - 24883824 \cdot 2^{-24} + 1993950 \cdot 2^{-20} + 17315 \cdot 2^{-16} + 15 \cdot 2^{-12}) \cdot 2^{3n} \Big] \\
&= 2^{8n} + 15345 \cdot 2^{6n} + 107310 \cdot 2^{5n} + 37128000 \cdot 2^{4n} + 329001120 \cdot 2^{3n} \\
&+67088385 \cdot 2^{8} \cdot 2^{2n} + 26043255 \cdot 2^{12} \cdot 2^{n} + 2^{16} \cdot 14881860.
\end{aligned}$$

The case n=1:

$R_{4,1}^{(10)} = 2^{8} + 15345\cdot 2^{6} + 107310\cdot 2^{5} + 37128000\cdot 2^{4} + 329001120\cdot 2^{3}$
$+ 67088385\cdot 2^{8}\cdot 2^{2} + 26043255\cdot 2^{12}\cdot 2 + 2^{16}\cdot 14881860 = 587\cdot 2^{31}$
Equally we obtain :
$R_{4,1}^{(10)} = 2^{37}\sum_{i=0}^{2}\Gamma_{i}^{2\times 10}2^{-i4} = 2^{37}\cdot[1+3\cdot 2^{-4}+2044\cdot 2^{-8}] = 587\cdot 2^{31}$ see [3],[4]

The case n=2:

$R_{4,2}^{(10)} = 2^{16} + 15345\cdot 2^{12} + 107310\cdot 2^{10} + 37128000\cdot 2^{8} + 329001120\cdot 2^{6}$
$+ 67088385\cdot 2^{8}\cdot 2^{4} + 26043255\cdot 2^{12}\cdot 2^{2} + 2^{16}\cdot 14881860 = 6361\cdot 2^{28}$
Equally we obtain :
$R_{4,2}^{(10)} = 2^{34}\sum_{i=0}^{4}\Gamma_{i}^{\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]\times 10}2^{-i4} = 2^{34}\cdot[1+9\cdot 2^{-4}+6174\cdot 2^{-8}+42840\cdot 2^{-12}+4145280\cdot 2^{-16}] = 6361\cdot 2^{28}$ see [5]

The case n=3:

$R_{4,3}^{(10)} = 2^{24} + 15345\cdot 2^{18} + 107310\cdot 2^{15} + 37128000\cdot 2^{12} + 329001120\cdot 2^{9}$
$+ 67088385\cdot 2^{8}\cdot 2^{6} + 26043255\cdot 2^{12}\cdot 2^{3} + 2^{16}\cdot 14881860 = 1552553\cdot 2^{21}$
Equally we obtain :
$R_{4,3}^{(10)} = 2^{31}\sum_{i=0}^{6}\Gamma_{i}^{\left[\begin{smallmatrix}2\\2\\2\end{smallmatrix}\right]\times 10}2^{-i4} = 2^{31}\cdot[1+21\cdot 2^{-4}+14602\cdot 2^{-8}+302400\cdot 2^{-12}+32004000\cdot 2^{-16}+430133760\cdot 2^{-20}+8127479808\cdot 2^{-24}] = 1552553\cdot 2^{21}$ see [6]

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